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Research article

A mathematical study of heat and mass transfer on non-linear MHD boundary layer flow

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Abstract

This paper demonstrates the mathematical analysis of an MHD viscous flow due to a shrinking sheet in the presence of suction. The governing unsteady boundary layer equations are reduced to dimensionless form by similarity transformations. These transformed simultaneous ordinary differential equations are solved analytically. The approximate analytical expressions of the dimensionless velocity, dimensionless temperature and dimensionless concentration are derived by using the Homotopy analysis method. Further the graphical representations of the all the above dimensionless quantities for all values of the other dimensionless parameters are investigated. The Homotopy analysis method can be easily extend it to solve other non-linear MHD fluid flow problems in engineering and applied sciences.

Keywords: Unsteady flow; Chemical reaction; Shrinking sheet; Heat transfer; Magnetic effect; Homotopy analysis method

1. Introduction

The flow and heat transfer of a viscous and incompressible fluid [2] caused by a continuously moving or stretching surface is pertinent to many manufacturing processes. The behavior of heat transfer and flow over a shrinking surface, an outlook of mechanical engineering and chemical engineering, deals with applications in industries such as the wire drawing, hot rolling and glass wire production. A number of technical processes concerning polymers involve the cooling of continuous strips or filaments by drawing them through a quiescent fluid. Further, glass blowing, continuous casting of metals and spinning of fibers involve the flow due to a stretching surface. In chemical and hydrometallurgical industries, the study of heat and mass transfer with magnetic effect is of considerable importance. In various situations, many authors Ramasamy Kandasamy et.al [1], Bhattacharyya et.al [7], Crane [6], Mohammadeian et.al [3], Brady [5], Schlitching[8] have discussed the effect of heat and mass transfer on nonlinear MHD boundary layer flow and Gupta et.al [4] depicted the heat and mass transfer on a stretching sheet with suction and blowing. Magnetohydrodynamic (MHD) is the study of magnetic properties of electrically conducting fluids and is important in most of the areas of science and engineering such as MHD power generation, MHD flow generators and MHD pumps. In recent times, the MHD fluid flow is important in many processes including drying evaporation at the surface of a water body, energy transfer in a wet cooling tower, flow in a desert cooler, generating electric power, food processing, groves of fruit trees and crop damage because of freezing. There is always a molecular diffusion of species on practical diffusive operations in the presence of chemical reaction within or at the boundary. There are two types of reactions namely, homogeneous, heterogeneous reaction exist in a restricted region or within the boundary of a phase. Formation of smog is an important example of a first-order homogeneous chemical reaction. Several researchers have discussed the facts of flows with chemical reactions. We can have large number of articles in the literature concerning different problems for Newtonian and non-Newtonian fluids with or without heat transfer analysis, for example, Hayat et.al [9] gave the analytic solution of MHD flow of a second grade fluid over a shrinking sheet Wang [11]presented unsteady shrinking film solution, Usha et.al [12] showed the axisymmetric motion of a liquid film on an unsteady stretching surface, Sajid et.al [13] showed the rotating flow of a viscous fluid over a shrinking surfaceand the existence and uniqueness of steady viscous hydrodynamic flow due to a shrinking sheet in the presence of suction have been proved by Miklavcic et.al [10]. The purpose of this study is obtaining an analytical solution on the effect of chemical reaction, heat and mass transfer on nonlinear MHD boundary layer past a porous shrinking sheet in the presence of suction.

2. Mathematical Formulation of the problem

The MHD flow of an incompressible viscous fluid over a shrinking sheet at $y=0$ is taken. $x-axis$ is along and $y-axis$ is perpendicularly depicted in the sheet respectively, as shown in Fig.1. The fluid is assumed to be Newtonian and electrically conducting and the flow is confined to $y>0$. A constant magnetic field of strength B_0 acts in the direction of $y-axis$. The induced magnetic field is very small, which is a valid assumption on a laboratory scale. Our assumption is true when the magnetic Reynolds number is small. We can assume that the electric field $E=0$, as no electric field is applied and the effect of polarization of the ionized fluid is negligible.

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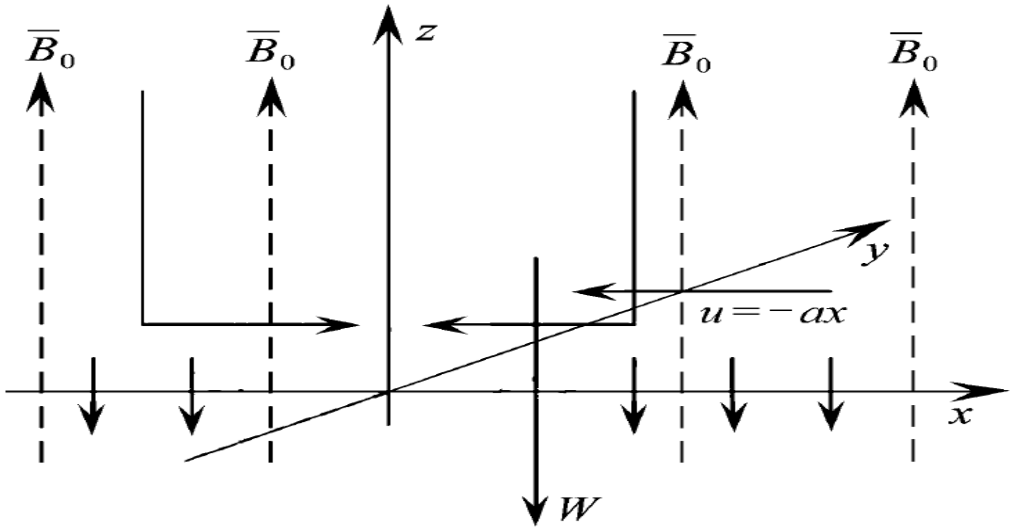

Fig. 1: Flow analysis on shrinking surface.

At the horizontal surface, a constant suction is established and the flow of chemical reactions are taken place, see Fig. 1. The momentum, energy and diffusion of the nonlinear boundary layer equations for the MHD flow in terms of vector notation are defined as follows:

Continuity equation:

$$div\vec{V} = 0 \tag{1}$$

Momentum equation:

$$(\vec{V}.grad\,\vec{V}) = -\frac{1}{\rho}grad\,\rho + v\nabla^2\vec{V} + \frac{1}{\rho}\vec{j}\times\vec{B} \tag{2}$$

Energy equation:

$$(\vec{V}.grad)T = \frac{k_e}{\rho c_p}\nabla^2 T \tag{3}$$

Species concentration equation: (4)

Where $\vec{j} = \sigma(\vec{E} + \vec{V}\times\vec{B} - \frac{1}{en_e}\vec{j}\times\vec{B} - \frac{1}{en_e}grad p_e div\,\vec{B}) = 0, curl\,\vec{H} = 0\,and\,curl\,\vec{E} = 0$, here $\vec{V}$ is the velocity vector, p is the pressure, v is the kinematic coefficient of viscosity.

Continuity equation in terms of vector notation (unsteady flow) is $\frac{\partial\rho}{\partial t} + \nabla.(\rho\vec{V}) = 0$. For steady incompressible flow: $\frac{\partial\rho}{\partial t} = 0\,and\,\rho$ is a constant. Continuity equation becomes $\nabla.(\rho\vec{V}) = 0$, it implies that $\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$. Finally, the continuity equation (steady flow) is reduced to $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$.

The governing boundary layer equations of momentum, energy and diffusion for mixed convection flow neglecting Joule's viscous dissipation can be simplified using the above equations to the following equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \;\; (continuity) \tag{5}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{1}{\rho}\frac{\partial \rho}{\partial x}+v\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{\sigma B_0^2}{\rho}u-\frac{v}{K}u \;\; (x-Momentum) \tag{6}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\frac{1}{\rho}\frac{\partial \rho}{\partial y}+v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{\sigma B_0^2}{\rho}v-\frac{v}{K}v \;\; (y-Momentum) \tag{7}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\frac{1}{\rho}\frac{\partial \rho}{\partial z}+v\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \;\; (z-Momentum) \tag{8}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right) \;\; (Energy) \tag{9}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=D\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}+\frac{\partial^2 C}{\partial z^2}\right)-k_1 C \;\; (Diffusion) \tag{10}$$

Here u,v,w denotes the velocity components in the x, y and z directions respectively, v is the kinematic viscosity, p is the pressure, σ is the electrical conductivity, ρ is the density of the fluid, B_0 is the magnetic induction, α is the thermal conductivity of the fluid, μ is the dynamic viscosity, K is the porous medium permeability, $c_p g$ is the specific heat at constant pressure and k_1 is the rate of chemical reaction.

The boundary conditions applicable to the present flow are

$$\left.\begin{array}{l} u=-U=-ax, v=-a(m-1)y, \\ w=-W, T=T_w C=C_w \;\; at\; y=0, \\ u\to 0, T\to T_\infty, C\to C_\infty \;\; as\; y\to\infty \end{array}\right\} \tag{11}$$

Here $a>0$ denotes the shrinking constant, $W>0$ denotes the suction velocity. When $m=1$, sheet shrinks only in $x-direction$ and when $m=2$, the sheet shrinks axisymmetrically.

Introducing the following similarity transformations

$$\left.\begin{array}{l} u=axf'(\eta), v=a(m-1)yf'(\eta), w=-\sqrt{avm}\, f(\eta), \eta=\sqrt{\frac{a}{v}}z, \\ \theta=\frac{T-T_\infty}{T_w-T_\infty} \;\; and \;\; \phi=\frac{C-C_\infty}{C_w-C_\infty} \end{array}\right\} \tag{12}$$

Equation (1) is identically satisfied and the eqn. (8) can be integrated to give

$$\frac{p}{\rho}-v\frac{\partial w}{\partial z}-\frac{w^2}{2}+constant \tag{13}$$

The eqns. (6) - (11) reduces to the following boundary value problem

$$f''' - (M^2 + Pr\lambda) f' - f'^2 + m\, f\, f'' = 0 \tag{14}$$

$$\theta'' + m\, Pr\, f\, \theta' - Pr\, \theta\, f' = 0 \tag{15}$$

$$\phi'' - Sc\, f'\phi + m\, Sc\, f\phi' - Sc\,\gamma\,\phi = 0 \tag{16}$$

The boundary conditions can be written as

$$\left.\begin{array}{l} \eta = 0 : \; f(0) = S, f'(0) = -1, \theta(0) = 1, \phi(0) = 1 \\ \eta \to \infty, \; f'(\infty) = 0, \theta(\infty) = 0, \phi(\infty) = 0 \end{array}\right\} \tag{17}$$

In which Pr is the Prandtl number, Sc is the Schmidt number, M^2 is the Magnetic parameter, γ is the Chemical reaction parameter, λ is the porosity parameter and S is the suction parameter can be defined as follows:

$$pr = \frac{v}{\alpha}, Sc = \frac{v}{D}, M^2 = \frac{\sigma B_0^2}{\rho a}, \gamma = \frac{k_1}{a}, \lambda = \frac{\alpha}{a\mathrm{K}} and\; S = \frac{W}{m\sqrt{av}} \tag{18}$$

The sheet shrinks in *x*-direction, if $m = 1$ and the sheet shrinks in axisymmetrically, if $m = 2$.

The mass diffusion equation (16) can be altered to meet these conditions if we take $\gamma > 0$ for destructive reaction, $\gamma = 0$ for no reaction and $\gamma < 0$ for generative reaction.

3. Solution of the problem using the Homotopy Analysis Method:

HAM is a non-perturbative analytical method for obtaining series solutions to nonlinear equations and has been successfully applied to numerous problems in science and engineering [14-29]. In comparison with other perturbative and non-perturbative analytical methods, HAM offers the ability to adjust and control the convergence of a solution via the so-called convergence-control parameter. Because of this, HAM has proved to be the most effective method for obtaining analytical solutions to highly non-linear differential equations. Previous applications of HAM have mainly focused on non-linear differential equations in which the non-linearity is a polynomial in terms of the unknown function and its derivatives. As seen above, the non-linearity present in electro hydrodynamic flow takes the form of a rational function, and thus, poses a greater challenge with respect to finding approximate solutions analytically. Our results show that even in this case, HAM yields excellent results.

Liao [14-22] proposed a powerful analytical method for non-linear problems, namely the Homotopy analysis method. This method provides an analytical solution in terms of an infinite power series. However, there is a practical need to evaluate this solution and to obtain numerical values from the infinite power series. In order to investigate the accuracy of the Homotopy analysis method (HAM) solution with a finite number of terms, the system of differential equations were solved. The Homotopy analysis method is a good technique comparing to another perturbation method. The Homotopy analysis method contains the auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region of solution series. The approximate analytical solution of the eqns. (14)-(18) using the Homotopy analysis method [37] is

$$f(\eta)=\left(S+\frac{e^{-Pr\,MSmx}}{Pr\,MSm}-\frac{1}{Pr\,MSm}\right)-h\left[\begin{array}{l}-\dfrac{e^{-Pr\,MSmx}(-M^2-Pr\,\lambda-m^2S^2\,Pr\,M+m)}{(Pr\,MSm)^3}\\ -\dfrac{-e^{-2\,Pr\,MSmx}(1-m)}{(2\,Pr\,MSm)^3}\\ -\left(\begin{array}{l}\dfrac{-M^2-Pr\,\lambda-m^2S^2\,Pr\,M+m}{(Pr\,MSm)^2}\\ +\dfrac{(1-m)}{(2\,Pr\,MSm)^2}\end{array}\right)x\\ +\left(\dfrac{-M^2-Pr\,\lambda-m^2S^2\,Pr\,M+m}{(Pr\,MSm)^3}+\dfrac{(1-m)}{(2\,Pr\,MSm)^3}\right)\end{array}\right] \tag{19}$$

The corresponding dimensionless velocity $f'(\eta)$ using the eqn. (19) is given by

$$f'(\eta)=\left(-e^{-Pr\,MSmx}\right)-h\left[\begin{array}{l}\dfrac{e^{-Pr\,MSmx}(-M^2-Pr\,\lambda-m^2S^2\,Pr\,M+m)}{(Pr\,MSm)^2}\\ -\dfrac{-e^{-2\,Pr\,MSmx}(1-m)}{(2\,Pr\,MSm)^2}\\ -\left(\dfrac{-M^2-Pr\,\lambda-m^2S^2\,Pr\,M+m}{(Pr\,MSm)^2}+\dfrac{(1-m)}{(2\,Pr\,MSm)^2}\right)\end{array}\right] \tag{20}$$

The dimensionless temperature is given by

$$\theta(\eta)=\left(e^{-ScmSx}\right)-h\left[\begin{array}{l}\dfrac{S^2m^2\,Pr\,Sce^{-ScmSx}}{(ScmS)^2}+\dfrac{Sm^2\,Pr\,Sce^{-(\lambda\,Pr\,M+ScmS)x}}{(\lambda\,Pr\,M)(\lambda\,Pr\,M+Sc)^2}-\dfrac{Sm^2\,Pr\,Sce^{-ScmSx}}{\lambda\,Pr\,MSc^2m^3S^2}\\ -\dfrac{Pr\,e^{-(\lambda\,Pr\,MmS+ScmS)x}}{(\lambda\,Pr\,MmS+Sc)^2}\\ -\dfrac{S^2m^2\,Pr\,Sc}{(ScmS)^2}+\dfrac{Sm^2\,Pr\,Sc}{(\lambda\,Pr\,M)(\lambda\,Pr\,M+Sc)^2}-\dfrac{Sm^2\,Pr\,Sc}{\lambda\,Pr\,MSc^2m^3S^2}\\ -\dfrac{Pr}{(\lambda\,Pr\,MmS+Sc)^2}\end{array}\right] \tag{21}$$

The dimensionless concentration is given by

$$\phi(\eta)=\left(e^{-\lambda\gamma x}\right)-h\left[Sc\left(\begin{array}{l}\dfrac{-e^{-(Pr\,MmS+\lambda\gamma)x}}{(Pr\,MmS+\lambda\gamma)^2}+\dfrac{mSe^{-\lambda\gamma x}}{\lambda\gamma}\\ -\dfrac{me^{-\lambda\gamma x}}{\lambda\gamma\,Pr\,M}\\ +\dfrac{m\lambda\gamma e^{-(Pr\,MSm+\lambda\gamma)x}}{Pr\,MSm(Pr\,MSm+\lambda\gamma)^2}\\ +\dfrac{e^{-\lambda\gamma x}}{\lambda^2\gamma}\end{array}\right)-Sc\left(\begin{array}{l}\dfrac{-1}{(Pr\,MmS+\lambda\gamma)^2}+\dfrac{mS}{\lambda\gamma}\\ -\dfrac{m}{\lambda\gamma\,Pr\,M}+\dfrac{1}{\lambda^2\gamma}\\ +\dfrac{m\lambda\gamma}{Pr\,MSm(Pr\,MSm+\lambda\gamma)^2}\end{array}\right)\right] \tag{22}$$

4. Results and Discussion

In this section the effects of chemical reaction in the presence of Magnetic parameter M^2, Prandtl number Pr, Schmidt number Sc, chemical reaction parameter γ, Porosity parameter λ and Suction parameter S, Skin friction m will be discussed. Fig. 1 illustrates the schematic diagram of a flow analysis on shrinking sheet. Fig. 2 depicts the dimensionless velocity $f'(\eta)$ w.r.to the similarity variable η. Here we infer that the velocity profile $f'(\eta)$ increases with increase in the suction parameter S in some fixed values of other dimensionless parameters Pr,λ,m,M^2. Fig. 3 to 5 shows the dimensionless velocity $f'(\eta)$, dimensionless concentration $\phi(\eta)$, dimensionless temperature $\theta(\eta)$ w.r.to the similarity variable η. From Fig. 3 it is observed that the velocity and temperature profile increases with the increase in the reaction parameter $(\gamma > 0)$ whereas the concentration of the fluid decreases with $\gamma > 0$ and in some fixed values of other parameter Pr,λ,Sc,m,M^2,S.

From Fig. 4 it is noted that the velocity of the fluid increases and the temperature and concentration of the fluid decreases with increase in the magnetic parameter M^2 in some fixed values of the other parameter $Pr,\lambda,Sc,m, S,\gamma$. From Fig. 5 it is noted that when the skin friction m increases, the corresponding dimensionless velocity increases whereas the dimensionless temperature and concentration profiles decreases in some fixed values of the other dimensionless parameters $Pr,\lambda,Sc,\gamma,M^2,S$. Table 1 shows the analysis for skin friction, rate of heat and mass transfer.

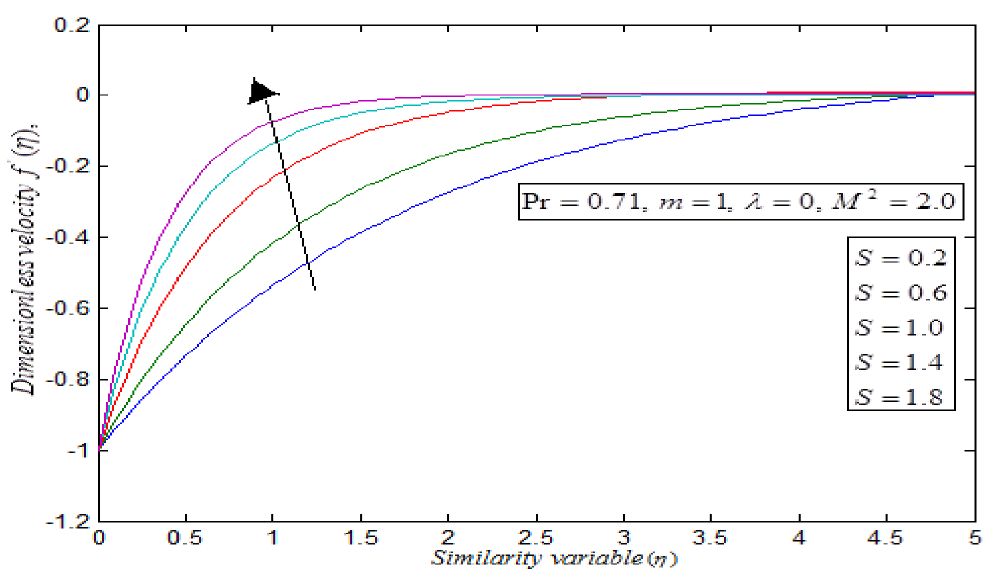

Fig. 2: Dimensionless velocity $f'(\eta)$ versus the Similarity variable η. The curves are plotted for various values of the suction parameter S and some fixed values of the other parameter Pr,λ,m,M^2 using the eqn. (20), when $h = -0.0067$.

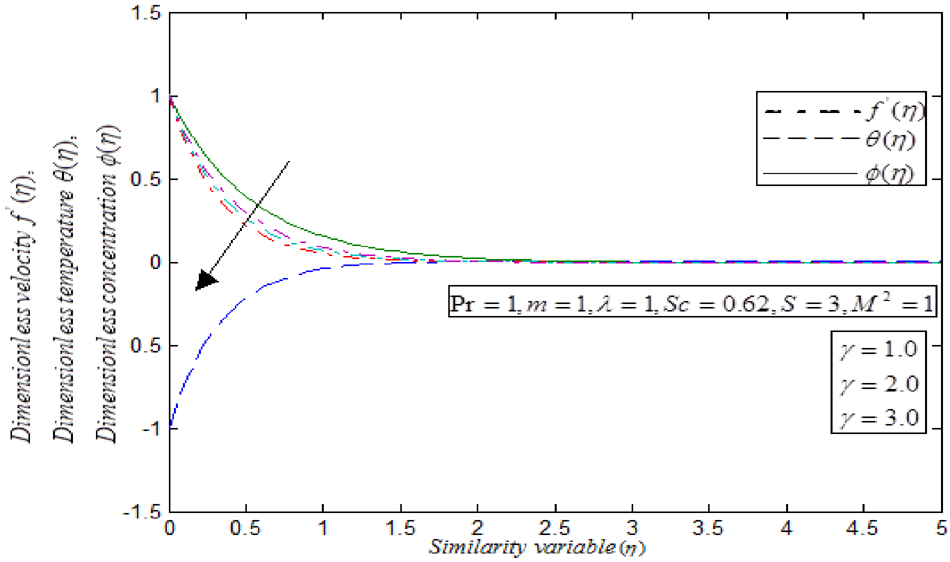

Fig. 3: Dimensionless velocity $f'(\eta)$, dimensionless temperature $\theta(\eta)$ and dimensionless concentration $\phi(\eta)$ versus the Similarity variable η. The curves are plotted for various values of the chemical reaction parameter γ and some fixed values of the other parameter $Pr, \lambda, Sc, S, m, M^2$ using the eqns. (20), (21) and (22), when $h = -0.00362$.

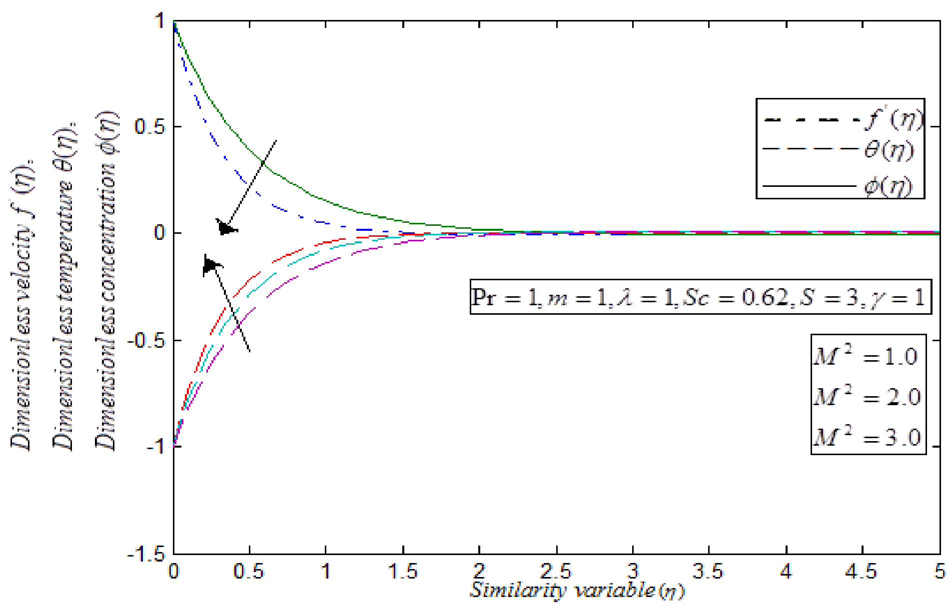

Fig. 4: Dimensionless velocity $f'(\eta)$, dimensionless temperature $\theta(\eta)$ and dimensionless concentration $\phi(\eta)$ versus the Similarity variable η. The curves are plotted for various values of the magnetic parameter M^2 and some fixed values of the other parameter $Pr, \lambda, Sc, m, S, \gamma$ using the eqns. (20), (21) and (22), when $h = -0.0066$.

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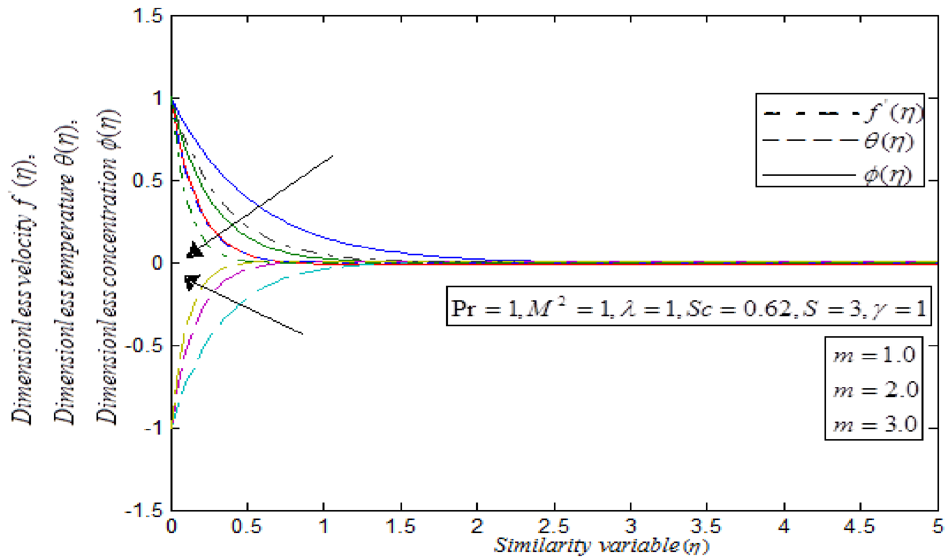

Fig. 5: Dimensionless velocity $f'(\eta)$, dimensionless temperature $\theta(\eta)$ and dimensionless concentration $\phi(\eta)$ versus the Similarity variable η. The curves are plotted for various values of the skin friction m and some fixed values of the other parameter $Pr, \lambda, Sc, M^2, S, \gamma$ using the eqns. (20), (21) and (22), when $h = -0.0066$

Table 1:

Analysis for skin friction, rate of heat and mass transfer.

$f'''(0)$	$\theta'(0)$	$\phi'(0)$		Parameter
3.302667	-2.664367	-2.410803	$\lambda = 1.0$	Porosity
3.561000	-2.681655	-2.410531	$\lambda = 2.0$	
4.001000	-2.702625	-2.426233	$\lambda = 3.0$	
3.300000	-2.662428	-2.411060	$\gamma = 1.0$	Chemical reaction parameter
3.300000	-2.662428	-2.825848	$\gamma = 2.0$	
3.300000	-2.662428	-3.344100	$\gamma = 3.0$	
3.300000	-2.662428	-2.000000	$M^2 = 1.0$	Magnetic strength
3.357889	-2.680901	-2.332000	$M^2 = 2.0$	
3.677377	-2.699117	-2.498500	$M^2 = 3.0$	
2.413200	-1.862588	-1.818597	$m = 1.0$	Shrinking
4.148410	-3.725691	-2.463586	$m = 2.0$	
6.000000	-5.588726	-3.744444	$m = 3.0$	

5. Conclusion

The present study provides similarity solutions on the effect of chemical reaction, heat and mass transfer for the non-linear MHD boundary layer flow past a porous shrinking sheet in the presence of suction. The approximated analytical expressions of the dimensionless velocity, dimensionless temperature and dimensionless concentration are derived for all values of the other dimensionless parameters by using the Homotopy analysis method. This study is used to observe the movement of oil or gas and water through the reservoir of an oil or gas field, in the migration of underground water and in the filtration and water purification processes. The results of the problem are also of great interest in geophysics in the study of interaction of the geomagnetic field with the fluid in the geothermal region.

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Appendix: A

Basic concept of the Homotopy analysis method [14-22]

Consider the following differential equation:

$$N[u(t)] = 0 \tag{A.1}$$

Where N is a non-linear operator, t denote an independent variable, $u(t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the conventional Homotopy method, Liao (2012) constructed the so-called zero-order deformation equation as:

$$(1-p)L[\varphi(t;p)-u_0(t)] = phH(t)N[\varphi(t;p)] \tag{A.2}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L an auxiliary linear operator, u_0 (t) is an initial guess of $u(t)$, $\varphi(t:p)$ is an unknown function. It is important to note that one has great freedom to choose auxiliary unknowns in HAM. Obviously, when $p=0$ and $p=1$, it holds:

$$\varphi(t;0) = u_0(t) \text{ and } \varphi(t;1) = u(t) \text{ respectively.} \tag{A.3}$$

Thus, as p increases from 0 to 1, the solution $\varphi(t;p)$ varies from the initial guess $u_0(t)$ to the solution u (t). Expanding $\varphi(t;p)$ in Taylor series with respect to p, we have:

$$\varphi(t;p) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t) p^m \quad (A.4)$$

Where

$$u_m(t) = \frac{1}{m!} \frac{\partial^m \varphi(t;p)}{\partial p^m}\Big|_{p=0} \quad (A.5)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter *h*, and the auxiliary function are so properly chosen, the series (A.4) converges at *p* =1 then we have:

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t). \quad (A.6)$$

Differentiating (A.2) for *m* times with respect to the embedding parameter *p*, and then setting *p* = 0 and finally dividing them by *m*!, we will have the so-called *mth* -order deformation equation as:

$$L[u_m - \chi_m u_{m-1}] = hH(t)\Re_m(\vec{u}_{m-1}) \quad (A.7)$$

Where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(t;p)]}{\partial p^{m-1}} \quad (A.8)$$

And

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \quad (A.9)$$

Applying L^{-1} on both side of equation (A7), we get

$$u_m(t) = \chi_m u_{m-1}(t) + hL^{-1}[H(t)\Re_m(\vec{u}_{m-1})] \quad (A10)$$

In this way, it is easily to obtain u_m for $m \ge 1$, at M^{th} order, we have

$$u(t) = \sum_{m=0}^{M} u_m(t) \quad (A.11)$$

When $M \to +\infty$, we get an accurate approximation of the original equation (A.1). For the convergence of the above method we refer the reader to Liao [20]. If equation (A.1) admits unique solution, then this method will produce the unique solution.

Appendix B:

Approximate analytical expressions of the non-linear differential eqns. (14)-(37) using the Homotopy analysis method

$$f''' - (M^2 + Pr\lambda) f' - f'^2 + m f f'' = 0 \quad (B.1)$$

$$\theta'' + m Pr f \theta' - Pr \theta f' = 0 \quad (B.2)$$

$$\phi'' - Sc f' \phi + m Sc f \phi' - Sc \gamma \phi = 0 \quad (B.3)$$

We construct the Homotopy for the eqns. (B.1), (B.2) and (B.3) are as follows:

$$(1-p)\left(f'''\right)-hp\left[f'''-(M^2+Pr\,\lambda)f'-f'^2+mff''\right]=0 \tag{B.4}$$

$$(1-p)\left(\theta''\right)-hp\left[\theta''+m\,Pr\,f\theta'-Pr\,\theta\,f'\right]=0 \tag{B.5}$$

$$(1-p)\left[\frac{d^2\theta}{d\xi^2}+\frac{Q+\psi\theta_s^4}{1-\beta\theta_a}\right]-hp\left[\frac{d^2\theta}{d\xi^2}+\frac{\frac{d^2\theta}{d\xi^2}\beta\theta}{1-\beta\theta_a}-\frac{\psi\theta^4}{1-\beta\theta_a}+\frac{Q+\psi\theta_s^4}{1-\beta\theta_a}\right]=0 \tag{B.6}$$

The approximate solution of the eqns. (B.4), (B.5) and (B.6) are as follows:

$$f=f_0+pf_1+p^2f_2+p^3f_3+\dots \tag{B.7}$$

$$\theta=\theta_0+p\theta_1+p^2\theta_2+p^3\theta_3+\dots \tag{B.8}$$

$$\phi=\phi_0+p\phi_1+p^2\phi_2+p^3\phi_3+\dots \tag{B.9}$$

Substituting the eqns. (B.7), (B.8) and (B.9) to the eqns. (B.4), (B.5) and (B.6) respectively, we get

$$(1-p)\left(\frac{d^3(f_0+pf_1+p^2f_2+..)}{d\eta^3}\right)=hp\left[\begin{array}{l}\dfrac{d^3(f_0+pf_1+p^2f_2+p^3f_3+\dots)}{d\eta^3}\\ -(M^2+Pr\,\lambda)\dfrac{d(f_0+pf_1+p^2f_2+p^3f_3+\dots)}{d\eta}\\ -\dfrac{d(f_0+pf_1+p^2f_2+p^3f_3+\dots)}{d\eta}^2\\ +m(f_0+pf_1+p^2f_2+..)\dfrac{d^2(f_0+pf_1+p^2f_2+..)}{d\eta^2}\end{array}\right] \tag{B.10}$$

$$(1-p)\left(\frac{d^2(\theta_0+p\theta_1+p^2\theta_2+\dots)}{d\eta^2}\right)=hp\left[\begin{array}{l}\dfrac{d^2(\theta_0+p\theta_1+p^2\theta_2+\dots)}{d\eta^2}\\ +m\,Pr\left(f_0+pf_1+\dots\right)\dfrac{d(\theta_0+p\theta_1+p^2\theta_2+\dots)}{d\eta}\\ -Pr(\theta_0+p\theta_1+p^2\theta_2+\dots)\dfrac{d\left(f_0+pf_1+\dots\right)}{d\eta}\end{array}\right] \tag{B.11}$$

$$(1-p)\left(\frac{d^2(\phi_0+p\phi_1+p^2\phi_2+\dots)}{d\eta^2}\right)=hp\left[\begin{array}{l}\dfrac{d^2(\phi_0+p\phi_1+p^2\phi_2+p^3\phi_3+\dots)}{d\eta^2}\\ -Sc\dfrac{d\left(f_0+pf_1+\dots\right)}{d\eta}(\phi_0+p\phi_1+p^2\phi_2+\dots)\\ +mSc\left(f_0+pf_1+\dots\right)\dfrac{d(\phi_0+p\phi_1+p^2\phi_2+\dots)}{d\eta}\\ -Sc\gamma(\phi_0+p\phi_1+p^2\phi_2+p^3\phi_3+\dots)\end{array}\right] \tag{B.12}$$

Comparing the coefficients of like powers of p in the eqns. (B.10), (B.11) and (B.12) we get

$$p^0 : \frac{d^3 f}{d\eta^3} = 0 \tag{B.13}$$

$$p^0 : \frac{d^2\theta}{d\eta^2} = 0 \tag{B.14}$$

$$p^0 : \frac{d^2\phi}{d\eta^2} = 0 \tag{B.15}$$

$$p^1 : \frac{d^3 f_1}{d\eta^3} - \frac{d^3 f_0}{d\eta^3} = h\left[-(M^2 + Pr\,\lambda)\frac{df_0}{d\eta} - \left(\frac{df_0}{d\eta}\right)^2 + mf_0 \frac{d^2 f_0}{d\eta^2} \right] \tag{B.16}$$

$$p^1 : \frac{d^2\theta_1}{d\eta^2} - \frac{d^2\theta_0}{d\eta^2} = h\left[\frac{d^2\theta_0}{d\eta^2} + m\,Pr\; f_0 \frac{d\theta_0}{d\eta} - Pr\,\theta_0 \right] \tag{B.17}$$

$$p^1 : \frac{d^2\phi_1}{d\eta^2} - \frac{d^2\phi_0}{d\eta^2} = h\left[-Scf_0{}'\phi_0 + mScf_0 \frac{d\phi_0}{d\eta} - Sc\gamma\phi_0 \right] \tag{B.18}$$

The initial approximations are as follows:

$$\left. \begin{array}{l} f_0(0) = S,\; f_0{}'(0) = -1,\; f_0{}'(\infty) = 0, \\ \theta_0(0) = 1,\; \theta_0(\infty) = 0, \\ \phi_0(0) = 1,\; \phi_0(\infty) = 0 \end{array} \right\} \tag{B.19}$$

$$\left. \begin{array}{l} f_i(0) = 0,\; f_i{}'(0) = 0,\; f_i{}'(\infty) = 0, \\ \theta_i(0) = 0,\; \theta_i(\infty) = 0, \\ \phi_i(0) = 0,\; \phi_i(\infty) = 0 \end{array} \right\} \quad where\; i = 1,2,3,\ldots\ldots. \tag{B.20}$$

For this HAM solution, we choose the initial guesses in the following form which satisfies the eqn. (B.19):

$$f_0(\eta) = S + \frac{e^{-Pr\,MSmx}}{Pr\,MSm} - \frac{1}{Pr\,MSm} \tag{B.21}$$

$$\theta_0(\eta) = \left(e^{-ScmSx}\right) \tag{B.22}$$

$$\phi_0(\eta) = \left(e^{-\lambda\gamma x}\right) \tag{B.23}$$

By solving the eqns. (B.16) - (B.18) using the boundary condition (B.20) we can obtain the following results:

$$f_1(\eta) = \left[\begin{array}{l} -\dfrac{e^{-Pr\,MSmx}(-M^2 - Pr\,\lambda - m^2S^2\,Pr\,M + m)}{(Pr\,MSm)^3} - \dfrac{-e^{-2\,Pr\,MSmx}(1-m)}{(2\,Pr\,MSm)^3} \\ -\left(\dfrac{-M^2 - Pr\,\lambda - m^2S^2\,Pr\,M + m}{(Pr\,MSm)^2} + \dfrac{(1-m)}{(2\,Pr\,MSm)^2} \right)x \\ +\left(\dfrac{-M^2 - Pr\,\lambda - m^2S^2\,Pr\,M + m}{(Pr\,MSm)^3} + \dfrac{(1-m)}{(2\,Pr\,MSm)^3} \right) \end{array} \right] \tag{B.24}$$

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$$\theta_1(\eta)=\begin{bmatrix} \dfrac{S^2m^2\,Pr\,Sce^{-ScmSx}}{(ScmS)^2}+\dfrac{Sm^2\,Pr\,Sce^{-(\lambda Pr M+ScmS)x}}{(\lambda Pr M)(\lambda Pr M+Sc)^2}-\dfrac{Sm^2\,Pr\,Sce^{-ScmSx}}{\lambda Pr MSc^2m^3S^2} \\ -\dfrac{Pr\,e^{-(\lambda Pr MmS+ScmS)x}}{(\lambda Pr MmS+Sc)^2} \\ -\dfrac{S^2m^2\,Pr\,Sc}{(ScmS)^2}+\dfrac{Sm^2\,Pr\,Sc}{(\lambda Pr M)(\lambda Pr M+Sc)^2}-\dfrac{Sm^2\,Pr\,Sc}{\lambda Pr MSc^2m^3S^2} \\ -\dfrac{Pr}{(\lambda Pr MmS+Sc)^2} \end{bmatrix} \tag{B.25}$$

$$\phi_1(\eta)=\left[Sc\begin{pmatrix} \dfrac{-e^{-(Pr MmS+\lambda\gamma)x}}{(Pr MmS+\lambda\gamma)^2}+\dfrac{mSe^{-\lambda\gamma x}}{\lambda\gamma} \\ -\dfrac{me^{-\lambda\gamma x}}{\lambda\gamma Pr M} \\ +\dfrac{m\lambda\gamma e^{-(Pr MSm+\lambda\gamma)x}}{Pr MSm(Pr MSm+\lambda\gamma)^2} \\ +\dfrac{e^{-\lambda\gamma x}}{\lambda^2\gamma} \end{pmatrix}-Sc\begin{pmatrix} \dfrac{-1}{(Pr MmS+\lambda\gamma)^2}+\dfrac{mS}{\lambda\gamma}-\dfrac{m}{\lambda\gamma Pr M} \\ +\dfrac{m\lambda\gamma}{Pr MSm(Pr MSm+\lambda\gamma)^2}+\dfrac{1}{\lambda^2\gamma} \end{pmatrix}\right] \tag{B.26}$$

According to the Homotopy analysis method we have

$$f=\lim_{p\to 1} f(\eta)=f_0+f_1 \tag{B.27}$$

$$\theta=\lim_{p\to 1}\theta(\eta)=\theta_0+\theta_1 \tag{B.28}$$

$$\phi=\lim_{p\to 1}\phi(\eta)=\phi_0+\phi_1 \tag{B.29}$$

Using the eqns. (B.21) - (B.26) in (B.27) – (B.29) respectively, we obtain the solutions in the text eqns. (19)–(22).

Appendix: C

Nomenclature

Symbol	Meaning
V	velocity
p	Pressure
v	Kinematic coefficient of viscosity
η	Similarity variable
ρ	density constant
$f'(\eta)$	Dimensionless velocity
θ	Dimensionless temperature
ϕ	Dimensionless concentration
Pr	Prandtl number

Sc	Schmidt number
M^2	Magnetic parameter
λ	Porosity parameter
γ	Chemical reaction parameter
m	Skin friction
S	Suction parameter