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Research Article

APPROXIMATION OF MULTI-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS BY AN ITERATIVE DECOMPOSITION METHOD

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Abstract

In this paper, an iterative decomposition method is applied to approximate the solution of multi-order fractional differential equations by Caputo form of the fractional derivatives. The method presents solutions as rapidly convergent infinite series of easily computable terms. Results obtained are compared favourably with known results in terms of accuracy and efficiency.

Keywords and Phrases: Multi-order fractional differential equations, Iterative decomposition method, Convergent infinite series.

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1. Introduction

Mathematical modelling of physical phenomena result in functional equations which may be Ordinary Differential , Partial Differential , Stochastic Differential , Difference, Integro – Differential, Difference - Differential and so on.In many models which involve differential equations , there may be more than one derivative of the dependent variable with respect to independent variable , appearing in such equations.The use of fractional differential and integral operators in mathematical models has become increasing widespread in recent years. Several forms of fractional differential equations have been propose in standard models , with significant interest in developing appropriate numerical solution techniques. However, most of the work published have been concerned with single derivatives equations. In this work, we focus on providing solutions to multi – term fractional differential equations , where the highest order derivative may be greater than one.

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Multi – term fractional differential equations or composite fractional equations have been known to be used in the modelling of visco – elasticity, electro – chemistry and material science [1, 4] . Very well – known examples of multi – term fractional differential equations include the Basset equation [1, 4, 6] and the Bagley – Torvick equation [4] .

One very great constraint ti the use of multi – term fractional differential equation in models is the existence of limited numerical schemes for their solutions.Owing to the numerous applications of fractional differential equations in diverse fields, the solution techniques for fractional differential equations continue to attract renewed interest from researcher. A good number of researchers have proposed and applied some efficient approximation and analytical techniques for the solution of problems of fractional calculus. Such techniques have been applied to fractional differential equations, fractional integral equations, and in some cases fractional integro – differential equations.There have been attempts to solve multi – order fractional differential equations but a complete analysis has so far not been given [4,7].

Numerical methods for the solution of linear fractional differential equations involving only one fractional derivative are well established (see for example [4,8,9].In [6], a new algorithm for the numerical approximation of initial value problems for general linear multi – term fractional differential equations in the Caputo senses was discussed. The solution is obtained by applying the quadrature and the Laplace transform to the solution obtained in terms of the Mittag – Leffler functions. However, a large amount of computational effort is required.In [7], the solution of a linear multi – term fractional differential equation was obtained by reduction of the problem to a system of ordinary and fractional differential equations , each of order at most unity.

In [6], a method which decomposed the given fractional differential equation into a system of integer – order was considered for linear and nonlinear problems, while [9] considered the reformation of the Bagley – Torvik equation as a system of fractional differential equations of order $\frac{1}{2}$

It should be observed that most of the earlier methods mentioned convert multi – order fractional differential equation into systems of fractional differential equation of low orders, and any single term equation solver is then appliedMeanwhile, a good number of order researchers have applied variants or modifications of other numerical methods which have been tested on integer – order differential equations. Such include the Homotopy Analysis Method [8] , the Homotopy Perturbation Method [8,9] and the Adomian Decomposition Method.In this work, we applied the Iterative Decomposition Method which has been applied to integer – order differential equations, as well as fractional differential and fractional integro – differential equations. Unlike previous methods, which have been applied to approximate.

2. Problem Formulation.

We first reviewed some basic definitions of fractional calculus, which are essential for the study of problems of multi – term fractional differential equations. as well as fractional differential.

2.1 Definition 1. (Caputo Derivative):

The Caputo fractional derivative of $f(x)$ of order α>0 is defined as

$$D_*^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}} dt,\ n-1 < \alpha \leq n, n \in \mathbb{R} \qquad (1)$$

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2.2 Definition 2.

A function $f(x), x > 0$ is said to be in the space $C_{\mu}, \mu \epsilon \mathbb{R}$ if there exists $p \in \mathbb{R}, p > \mu$ such that $f(x0 = x^{p} f_1(x)$, where $f_1(x) \in C[0, x]$. Clearly, $C_{\mu} \subset C_{\beta}\ if\ \beta \leq \mu$.

2.3 Definition 3.

A function $f(x), x > 0$ is said to be in the space $C_{\mu}^{m}, m \in \mathbb{N} \cup \{0\} if\ f^{(m)}(x) \in C_{\mu}$.

2.4 **Definition 2.4:** The Riemann-Liouvillefractional integral operator of order $\alpha \geq 0$, of a functionf $\in C_{\mu}, \ \mu \geq -1$ is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt,\ \alpha > 0,\ \alpha > 0, x > 0 \qquad (2)$$

Properties of the operator J^{α} can be found in [11] and include the following

$$J^{\alpha} f(x) = f(x)$$

$$J^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$$

$$J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x) \qquad (*)$$

$$J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x)$$

Also, if $m - 1 < \alpha < m, m \in \mathbb{N}\ and\ f \in C_{\mu}^{m}, \mu \geq -1, then$

$$J^{\alpha} f(x) = f(x) \qquad (3)$$

$$J^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{n=0}^{m-1} \frac{x^n}{n!} f^{(n)}(0) \ (4)$$

Multi – term fractional differential equation occur in various areas of Mechanics [8], example of which are the Baley – Torvik equations and the Baasset equations.

The general Multi – term fractional differential equation is of the form

$$D_*^{\alpha} y(x) = f(x, y(x), D_{\alpha}^{\beta_1} y(x), D_{\alpha}^{\beta_2} y(x), \dots, D_{\alpha}^{\beta_n} y(x) \qquad (5)$$

whereD_*^{α} is used to represent the Caputo – type fractional derivative .

We assumed that $\alpha > \beta_n > \beta_{n-1} > \dots > \beta_n$ and $\alpha - \beta_j \leq 1,\ \beta_j - \beta_{j-1} \leq 1$

for all j and $0 \leq \beta_1 < 1$

An initial value problem consists of (5) equipped with initial condition

$$y^k(0) = a_k\ ,\ k = 0.1 \dots\dots [\alpha] - 1 \qquad (6)$$

The notation $[\alpha]$ is used to denote the integer closest to , and not less than α .

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3. Methodology

For the multi – order fractional differential equation (5) by applying the operator

$J^{\alpha} = D^{-\alpha}$, the inverse of the operator D_*^{α} to both sides of equation (5), we have

$$y(x) = \sum_{k=0}^{m-1} \frac{x^k}{k!} y^k(0) + D^{-\alpha}\{f(x, y(x), D_*^{\beta_1} y(x), D_*^{\beta_2} y(x) \;.....D_*^{\beta_n} y(x)\} \qquad (7)$$

Thee Iterative Decomposition Method [10,11] suggests that (7) is then of the form

$$U = f + N(u) \qquad (8)$$

The solution is decomposed into the infinite series of convergent terms

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \qquad (9)$$

From (8), we have

$$N(u) = D^{-\alpha}\left\{f(x, y(x), D_*^{\beta_1} y(x),\, D_*^{\beta_2} y(x), \,......, D_*^{\beta_n} y(x)\right\} \qquad (10)$$

The operator N is then decomposed as

$$N(u) = N(u_0) + \{N(u_0+u_1) - N(u_0)\} + \{N(u_0+u_1+u_2) - N(u_0+u_1)\} + \ldots \qquad (11)$$

Let $G_0 = N(u_0)$

$$G_n = N\left(\sum_{i=0}^{n} u_i\right) - N\left(\sum_{i=0}^{n-1} u_i\right), \quad n = 1, 2, \ldots \qquad (12)$$

Then,

$$N(u) = \sum_{i=0}^{\infty} G_I$$

Set $u_0 = f$ (13)

From (8) and (9), we have

$$u_n = G_{n-1}, \qquad n = 1, 2, ... \qquad (14)$$

Then,

$$u = \sum_{n=0}^{N-1} y_n \qquad (15)$$

We can then approximate the solution (9) by

$$G_n = N\left(\sum_{i=0}^{n} u_i\right) - N\left(\sum_{i=0}^{n-1} u_i\right), \; n = 1, 2, \ldots \tag{16}$$

And,

$$\lim_{N\to\infty} \phi_N(x) = \lim_{n\to\infty} \sum_{n=0}^{N-1} y_n \tag{17}$$

3.1 CONVERGENCE OF ITERATIVE DECOMPOSITION METHOD.

From equation (12),

$$G_0 = N(u_0)$$

And in general,

$$G_n = N\left(\sum_{i=0}^{n} u_i\right) - N\left(\sum_{i=0}^{n-1} u_i\right), \; n = 1, 2, \ldots$$

It then follows that

$$G_1 = N(u_0 + u_1) - N(u_0) \tag{18}$$

Expanding $N(u_0 + u_1)$ in a Taylor series, we have

$$G_1 = N(u_0) + N'(u_0) + \frac{u^2}{2} N''(U_0) + \ldots + N(u_0)$$

$$= \sum_{k=1}^{\infty} \frac{u_1}{k!} N^{(k)}(u_0) \tag{19}$$

$$G_2 = N(u_0 + u_1 + u_2) + N \; (u_0 . + u_1)$$

$$= u_2 N'(u_0 + u_1) + \frac{u_2}{2!} N''(u_0 + u_1) + \ldots . \backslash$$

$$= \sum_{j=1}^{\infty} \left\{ \sum_{i=0}^{\infty} \frac{u^i}{i!} N^{(i+j)}(u_0) \right\} \frac{u_2^j}{j!} \tag{20}$$

$$G_3 = \sum_{i_3}^{\infty} \sum_{i_2}^{\infty} \sum_{i_1}^{\infty} \frac{u_3^{i_3}}{i_3} \frac{u_2^{i_2}}{i_2} \frac{u_1^{i_1}}{i_1} N^{(i_1+i_2+i_3)}(u_0) \tag{21}$$

In general,

$$G_n = \sum_{i_3}^{\infty}\sum_{i_2}^{\infty}\sum_{i_1}^{\infty} \quad \cdots \sum_{i_i}^{\infty}\left\{\left(\pi_{j=1}^{n} \frac{u_j^{ij}}{i_j\,!}\right) N^{\left(\sum_{K=1}^{N} i_K\right)}(u_0)\right\} \tag{22}$$

By the virtue of equation (8), to equation (21),

$$N(u) = G_0 + G_1 + G_2 + \ldots\ldots$$

$$= N(u_0) + \sum_{k=1}^{\infty} \frac{u_1^k}{k!} N^{(k)}(u_0) + \sum_{j=1}^{\infty}\left\{\sum_{i=0}^{\infty} \frac{u^i}{i!} N^{(i+j)}(u_0)\right\} \frac{u_2^j}{j!}$$

$$+ \sum_{i_3}^{\infty}\sum_{i_2}^{\infty}\sum_{i_1}^{\infty} \frac{u_3^{i_3}}{i_3\,!} \frac{u_2^{i_2}}{i_2\,!} \frac{u_1^{i_1}}{i_1!} N^{(i_{11}+i_2+i_3)}(u_0) + \ldots\ldots$$

$$= N(u_0) + N(u_1 + u_2 + u_3 + \ldots\ldots.)N'(u_0)$$

$$+\left[\frac{u_1^2}{2!} + \left(u_1u_2 + \frac{u_2^2}{2!}\right) + \left(u_3u_2 + u_3u_1 + \frac{u_3^2}{2!}\right) + \ldots\right] N^n(u_0)$$

$$+\left[\frac{u_1^3}{3!} + \left(u_2\frac{u_1^2}{2!} + \frac{u_2^2}{2!}u_1 + \frac{u_2^3}{3!}\right) + \ldots\right] N'''(u_0) + \ldots\ldots \tag{23}$$

From the equation in the sum of $N(u)$, we have

$$N(u) = N(u) + (u_1 + u_2 + u_3 + \ldots\ldots\_)N'(u_0) + \frac{(u_1 + u_2 + u_3 + \ldots)^2}{2!} N''(u_0)$$

$$\frac{(u_1 + u_2 + u_3 + \ldots\ldots)^3}{3!} N'''(u_0) + \ldots \tag{24}$$

From equation (*), it follows that equation (24) is a Taylor series expansion of

$N(u)$ about u_0.

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Numerical Examples

We now apply the method proposed in section 3 to some numerical examples, to establish the accuracy and efficiency of the method.

Example1.

Consider the initial value problem for the non-homogeneous Bagley-Torvik equation [9]

$$D_*^2\, y(x) + D_*^{\frac{3}{2}} y(x) + y(x) = 1 + x$$

$$y(0) = y'(0) = 0$$

The exact solution is $y(x) = 1 + x$

Example 2.

Consider the nonlinear multi-order fractional differential equation [5]

$$D_*^2 y(x) + D_*^{2.5} y(x) + y^2(x) = x^4$$

With the initial conditions

$$y(0) = y'(0) = 0, \quad y''(0) = 2$$

The exact solution is x^2

REMARK:

For the sake of comparison, example 1 has been solved by authors [4,5,6]. It is the Bagley - Torvik equation which models the motion of a rigid [plate in a Newtonian fluid.

Example 2 is a nonlinear multi-order fractional differential equation while example is the general Bessel equation.

SOLUTION TECHNIQUES

Example 4.2

Consider the nonlinear fractional integro-differential equation [9, 12]

$$D_*^{0.9} y(x) = -1 + \int_0^x y^2(t)dt, 0 \le x \le 1 \tag{20}$$

with initial condition $y(0) = 0$

Applying the inverse operator $D^{-0.9}$ to both sides of (20), we have

$$y(x) = D^{-0.9}(-1) + D^{-0.9}\{\int_0^x y^2(t)dt\} \tag{21}$$

$$= \frac{1}{\Gamma(0.9)}\int_0^x (x-t)^{0.1}dt + D^{-0.9}\{\int_0^x y^2(t)dt\}$$

$$= -1.008694635x^{0.9} + D^{-0.9}\{\int_0^x y^2(t)dt\} \quad (22)$$

Then, y(x) can be approximated as

$$y(x) = -1.039717197x^{0.9} + 0.1017155725x^{3.7} - 0.004345205712x^{5.5} \quad (23)$$

In table 3 below, we compare our result with the result for the same problem in [5]

Table 3

X	HPM [5]	IDM
0.0	0.0	0.0
0.125	-0.15997	-0.15996
0.25	-0.29790	-0.29798
0.375	-0.42689	-0.42690
0.5	-0.54824	-0.54844
0.625	-0.66086	-0.66193
0.75	-0.76327	-0.76788
s0.875	-0.85359	-0.85809
1.0	-0.92988	-0.92935

Example 4.3

Consider the initial value problem consisting the multi-fractional order integro-differential equation [7]

$$D_*^{0.5}y(x) = \frac{6}{\Gamma(5.5)}x^{2.5} - \frac{\Gamma(4)}{\Gamma(5.5)}x^{4.5} + J^{1.5}y(x),\ x\epsilon[0,1] \quad (24)$$

with initial condition $y(0) = 0$.

The exact solution of the problem is $y(x) = x^3$.

By applying the IDM to (24) in Example 4.3, we found that y(x) can be approximated as

$$y(x) = 1.00000445x^3 + (8.8623E - 5)x^5 - (2.3809E - 7)x^7$$

after only two iterations.

Table 4: Solution for Example 4.3

X	Exact Solution	IDM Approx.	Error
0.0	0.0	0.000000000	0,0
0.1	1.0E-3	1.000003584E-3	3.5637E-9
0.2	8.0E-3	8.000063956E-3	7.2376E-9
0.3	2.7E-3	2.700033545E-3	3.3545E-7
0.4	6.4E-2	0.06399937691	6.2309E-7
0.5	1.25E-1	0.1250033239	3.3239E-6
0.6	2.16E-1	0.2160078459	7.8459E-6
0.7	3.43E-1	0.3430164016	1.6402E-5
0.8	5.12E-1	0.5120312685	3.1268E-5
0.9	7.29E-1	0.7290554612	5.5461E-5
1.0	1.000000	1.000092835	9.2835E-5

4. Conclusion

From the examples given, the Iterative Decomposition Method (IDM) proved to be very efficient in the solution of fractional integro-differential equations. The solution of Example 4.1 by IDM is very close to the exact solution, even for very few terms of the approximating series. Example 4.2 shows that the method gives solutions that are comparable to known and tested methods, in terms of accuracy and efficiency. Furthermore, for cases where the exact solutions are unknown, the method is a useful tool for approximating solutions. The strength of the IDM is includes the fact that we do not require to find some polynomials, unlike in the case of the ADM. Neither do we require rigorous or elaborate mathematical details.

References

[1] Arikoglu, A., Ozkol, I. (2009), Solution of Fractional Integro-Differential Equations by using Fractional Differential Transfom Method, Chaos, Solitos and Fractals 40: 521-529

[2] Baleanu, D., Diethelm, K.,Scalas, E., Trujillo, J.J. (2012), Fractional Calculus: Models and Numerical Methods, World Scientific

[3] Diethelm, K. (1997), An Algorithm for the Numerical Solution of Differential Equations of Fractional Order, Elect. Trans. Num. Anal. 5: 1-6

[4] Ghasemi, M.M.(2007), Numerical Solution of the Non-Linear Integro-Differential Equations: Wavelet-Galerkin Method and Homotopy Perturbation Method, Appl. Math. Comp., 188: 450-455

[5] Ghazanfari, B., Ghazanfari, A.G., Veisi, F. (2010), Homotopy Perturbation Method for the Non-Linear Fractional Integro-Differential Equations, Aust. J. Basic Appl. Sci., 4(12): 5823-5829

[6] Ghorbani, A.(2008), Towards a new Analytical Method for solving Fractional Differential Equations, Comp. Math. Appl. Mech. Eng., 197: 4173-4177

[7] Hanan, I.K. (2011), The Homotopy Analysis Method for solving Multi-Fractional Order Integro-Differential Equations, J. Al-Nahrain Univ., 14(3): 139-143

[8] He, J.H. (1998), An Approximate Solution Technique Depending upon an Artificial Parameter, Comm. Nonlinear Sci. Num. Simul., 3(2): 92-97

[9] Mittal, R.C., Nigam, R.(2008), Solution of Fractional Integro-Differential Equations by Adomian Decomposition Method, Int. J. Appl. Math. Mech., 4: 87-94

[10] Momani, S., Odibat, Z. (2007), Numerical Approach to Differential Equations of Fractional Order, J. Comp. Appl. Math., 2007: 96-110

[11] Podlubny, I.(1999), Fractional Differential Equations, Academic Press, San Diego.

[12] Rawashdeh, E.A. (2006), Numerical Solution of Fractional Integro-Differential Equations by Collocation Method, Appl. Math. Comp., 176: 1-6

[13] Taiwo, O.A., Odetunde, O.S., Adekunle, Y.A. (2009), Numerical Approximation of One Dimensional Biharmonic Equations by an Iterative Decomposition Method, Int. J. Math. Sci. 20(1): 37-44